On a variational method for the Beltrami equations

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Abstract

We construct variations for the classes of regular solutions to degenerate Beltrami equations with restrictions of the set-theoretic type for the complex coefficient. On this basis, we prove the variational maximum principle and other necessary conditions of extremum.

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1 Introduction

Let D be a domain in $\mathbb{C}$, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The **Beltrami equation** in D has the form

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1.1}$$

where $\mu(z) : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., $f_{\overline{z}} = \overline{\partial} f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, f_x and f_y denote the partial derivatives of the mapping f in x and y, respectively. The function μ is the **complex coefficient** and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.2}$$

is the **dilatation quotient** or simply the **dilatation** of equation (1.1).

Recall that a mapping $f : D \to \mathbb{C}$ is called **regular at a point** $z_0 \in D$ if f has a total differential at the point and its Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 \neq 0$ (see, e.g., I.1.6 in [15]). A homeomorphism f of the class $W^{1,1}_{loc}$ is called **regular** if $J_f(z) > 0$ a.e. Finally, the **regular solution** of the Beltrami equation (1.1) in the domain D is a regular homeomorphism that satisfies (1.1) a.e. in D. The notion of the regular solution was first introduced in the paper [3].

A function $f : D \to \mathbb{C}$ is called **absolutely continuous on lines**, written $f \in$ ACL if for every closed rectangular R in D whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost every linear segment in R which is parallel to the sides of R (see, e.g., [1], p. 23).

Let $Q(z) : D \to I = [1, \infty]$ be an arbitrary function. A sense-preserving homeomorphism $f : D \to \mathbb{C}$ of the class ACL is called $Q(z)-$**quasiconformal** ($Q(z)-$q.c.) mapping if a.e.

$$K_{\mu_f}(z) := \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} \leq Q(z) \tag{1.3}$$

where $\mu_f = f_{\overline{z}}/f_z$ if $f_z \neq 0$ and $\mu_f = 0$ if $f_z = 0$. The function μ_f is called a complex characteristic and K_{μ_f} a dilatation of the mapping f.

Later $\mathbb{D} := \{\nu \in \mathbb{C} : |\nu| < 1\}$. Let $\mathcal{G}$ be the qroup of all linear-fractional mappings of $\mathbb{D}$ onto itself. A set M in $\mathbb{D}$ is called **invariant–convex** if all sets $g(M)$, $g \in \mathcal{G}$, are convex, see, e.g., [20], p. 636. In particular, such sets are convex. We say that a family of comact sets in $M(z) \subseteq \mathbb{D}$, $z \in \mathbb{C}$ is **measurable in the parameter** z, if for every closed set $M_0 \subseteq \mathbb{C}$ the set $E_0 = \{z \in \mathbb{C} : M(z) \subseteq M_0\}$ is measurable by Lebesgue (cf., e.g., [25]). Later we use the following notations

$$Q_M(z) := \frac{1 + q_M(z)}{1 - q_M(z)}, \quad q_M(z) := \max_{\nu \in M(z)} |\nu| . \tag{1.4}$$

Let $M(z)$, $z \in \mathbb{C}$ be a family of compact sets in $\mathbb{D}$ measurable in the parameter z. Let us denote by $\mathfrak{M}_M$ the class of all measurable functions satisfying the condition $\mu(z) \in M(z)$ a.e., and by H_M^* the collection of all regular homeomorphisms $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with the complex characteristics in $\mathfrak{M}_M$ and the normalization $f(0) = 0,\ f(1) = 1,\ f(\infty) = \infty$. In the previous papers [16] and [21], it was proved a series of criteria for the compactness of the classes H_M^* under the corresponding conditions on the function Q_M, cf. also [24], for invariant-convex $M(z)$, $z \in \mathbb{C}$. Note that the last condition implies convexity of the set of the complex characteristics $\mathfrak{M}_M$. As we will see later, the last circumstance essentially simplifies the construction of variations in the classes H_M^*.

One of the significant applications of compactness theorems is the theory of the variational method. The matter is that, in the compact classes, it is guaranteed the existence of extremal mappings for every continuous, in particular, nonlinear functionals. The variational method of the research of extremal problems for quasiconformal mappings was first applied by Belinskii P.P., see [2]. This method had a further development in papers of Gutlyanskii V.Ya., Krushkal' S.L., Kuhnau R., Ryazanov V.I., Schiffer M., Schober G. and others, see, e.g., [7]–[10], [12], [13], [23], [26], [27].

Recall that a mapping $f : X \to Y$ between metric spaces X and Y is called **Lipschitz** if $\mathrm{dist}(f(x_1),\, f(x_2)) \le M \cdot \mathrm{dist}(x_1,\, x_2)$ for some $M < \infty$ and for all $x_1,\, x_2 \in X$ where $\mathrm{dist}\,(x_1,\, x_2)$ denotes a distance in the metric spaces X and Y (see, e.g., [5], p. 63). The mapping f is called **be-Lipschitz** if in addition $M^{*}\cdot\mathrm{dist}(x_1,\, x_2) \le \mathrm{dist}(f(x_1),\, f(x_2))$ for some $M^{*} > 0$ and for all $x_1,\, x_2 \in X$.

2 Preliminaries

Let us give necessary facts from the theory of composition operators in Sobolev's spaces. Let D be a domain in the Euclidean space $\mathbb{R}^n$. Recall that the Sobolev space $L_p^1(D)$, $p \ge 1$, is the space of locally integrable functions $\varphi : D \to \mathbb{R}$ with the first partial generalized derivatives and with the seminorm

$$\|\varphi\|_{L_p^1(D)} = \|\bigtriangledown\varphi\|_{L_p(D)} = \left(\int\limits_D |\bigtriangledown\varphi|^p dm\right)^{1/p} < \infty \tag{2.1}$$

where m is the Lebesgue measure in $\mathbb{R}^n$, $\bigtriangledown\varphi$ is the **generalized gradient** of the function φ, $\bigtriangledown\varphi = \left(\frac{\partial\varphi}{\partial x_1}, ..., \frac{\partial\varphi}{\partial x_n}\right)$, $x = (x_1, ..., x_n)$, defined by the conditions

$$\int\limits_D \varphi \cdot \frac{\partial\eta}{\partial x_i}\, dm = -\int\limits_D \frac{\partial\varphi}{\partial x_i} \cdot \eta\, dm \qquad \forall\, \eta \in C_0^{\infty}(D),\ i = 1,\, 2, ..., n. \tag{2.2}$$

As usual, here $C_0^{\infty}(D)$ denotes the space of all infinitely smooth functions with a compact support in D. Similarly, they say that a vector-function belongs to the Sobolev class $L_p^1(D)$ if every its coordinate function belongs to $L_p^1(D)$. It is known the following fact, see [28] and [29].

Lemma 2.1. *Let f be a homeomorphism between domains D and D' in $\mathbb{R}^n$. Then the following statements are equivalent:*

1) the composition rule $f^{*}\varphi = \varphi \circ f$ generates the bounded operator

$$f^{*} : L_p^1(D') \to L_q^1(D),\ \ 1 \le q \le p < \infty\,, \tag{2.3}$$

2) the mapping f belongs to the class $W_{loc}^{1,1}(D)$ and the function

$$K_p(x,\, f) := \inf\left\{k(x) : |Df|(x) \le k(x)|J_f(x)|^{\frac{1}{p}}\right\} \tag{2.4}$$

belongs to $L_r(D)$ where r is defined from relation $1/r = 1/q - 1/p$.

In particular, for $n = 2$, $p = 2$ and $q = 1$, we have from here the following statement that will be useful for us.

Proposition 2.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a sense-preserving homeomorphism in the class $W^{1,1}_{loc}$ with $K_{\mu_f} \in L^1_{loc}$. Then $g \circ f \in W^{1,1}_{loc}$ for every mapping $g : \mathbb{C} \to \mathbb{C}$ in the class $W^{1,2}_{loc}$.*

As well-known, every quasiconformal mapping $g : \mathbb{C} \to \mathbb{C}$ belongs to the class $W^{1,2}_{loc}$, see, e.g., Theorem IV.1.2 in [15]. Thus, we come to the following conclusion.

Corollary 2.1. *For every quasiconformal mapping $g : \mathbb{C} \to \mathbb{C}$ and every sense-preserving homeomorphism $f : \mathbb{C} \to \mathbb{C}$ of the class $W^{1,1}_{loc}$ with $K_{\mu_f} \in L^1_{loc}$, the composition $g \circ f$ belongs to the class $W^{1,1}_{loc}$.*

The following statement on differentiability of the composition is proved similarly to Theorem 5.4.6 in [6].

Lemma 2.2. *Let f be a homeomorphism between domains D and D' in $\mathbb{R}^n$, the composition operator $f^* : L^1_p(D') \to L^1_q(D)$, $1 \le q \le p < \infty$, be bounded and let f has N^{-1}–property. Then for every function $\varphi \in L^1_p(D')$, a.e.*

$$\frac{\partial(\varphi \circ f)}{\partial x_i}(x) = \sum_{k=1}^{n} \frac{\partial \varphi}{\partial y_k}(f(x)) \cdot \frac{\partial f_k}{\partial x_i}(x), \ \ i = 1, ..., n. \tag{2.5}$$

Combining Lemmas 2.1 and 2.2, similarly to IC(1) in [1], we obtain.

Proposition 2.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a sense-preserving regular homeomorphism with $K_{\mu_f} \in L^1_{loc}$. Then, for every mapping $g : \mathbb{C} \to \mathbb{C}$ of the class $W^{1,2}_{loc}$, a.e.*

$$(g \circ f)_z = (g_w \circ f) f_z + (g_{\overline{w}} \circ f)\overline{f_{\overline{z}}}\,, \quad (g \circ f)_{\overline{z}} = (g_w \circ f) f_{\overline{z}} + (g_{\overline{w}} \circ f)\overline{f_z}\,. \tag{2.6}$$

Corollary 2.2. *In particular, formulas (2.6) hold for quasiconformal mappings $g : \mathbb{C} \to \mathbb{C}$.*

3 The construction of variations

This section is devoted to constructing variations in the classes H^*_M with a method whose idea was first proposed by Gutlyanskii V.Ya. in the paper [8] for analytic functions with a quasiconformal extension. Later, this approach was applied in [22]

under constraints for Q_M in measure of the exponential type.

Theorem 3.1. *Let $M(z)$, $z \in \mathbb{C}$ be an arbitrary family of convex sets in $\mathbb{D}$. Now, let $\mu \in \mathfrak{M}_M$ be a complex characteristic of a mapping $f \in H_M^*$ such that $K_\mu \in L^1_{loc}$ and $\nu \in \mathfrak{M}_M$ such that the function*

$$\varkappa = (\nu - \mu)/(1 - |\mu|^2) \tag{3.1}$$

belongs to the open unit ball in $L^\infty(\mathbb{C})$. Then there is a variation f_ε, $\varepsilon \in [0,\, 1/2]$ of the mapping f in the class H_M^* with the complex characteristic

$$\mu_\varepsilon = \mu + \varepsilon(\nu - \mu) = (1-\varepsilon)\mu + \varepsilon\nu\,, \quad \varepsilon \in [0,\, 1/2] \tag{3.2}$$

such that

$$f_\varepsilon(\zeta) = f(\zeta) - \frac{\varepsilon}{\pi}\int\limits_{\mathbb{C}} (\nu(z) - \mu(z))\, \varphi(f(z),\, f(\zeta))\, f_z^2\, dm_z + o(\varepsilon,\, \zeta) \tag{3.3}$$

where $o(\varepsilon\,, \zeta)/\varepsilon \to 0$ locally uniform with respect to $\zeta \in \mathbb{C}$ and

$$\varphi(w,\, w') = \frac{1}{w - w'} \cdot \frac{w'}{w} \cdot \frac{w'-1}{w-1}\,. \tag{3.4}$$

Proof. Denote by B a (Borel) set of all points $z \in \mathbb{C}$ where f has a total differential and $J_f(z) \neq 0$. Then by definition of the class H_M^* and by the Gehring–Lehto–Menshoff theorem $|\mathbb{C} \setminus B| = 0$ (see [18], cf. Theorem III.3.1 in [15]). Moreover, by Lemma 3.2.2 in [5] the set B can be splitted into a countable collection of sets B_l where f is bi-Lipschitz. By the Kirsbraun-McSchane theorem, see, e.g., Theorem 2.10.43 in [5], see also [11] and [17], the restrictions $f|_{B_l}$ admit extensions to Lipschitz mappings of $\mathbb{C}$. Thus, f has (N)–property on the set B and we may replace variables in integrals, see, e.g., Theorem 3.2.5 in [5]. Let

$$\varkappa_\varepsilon = \frac{\varepsilon\varkappa}{1 - \varepsilon\varkappa\overline{\mu}} = \varepsilon\varkappa\sum_{n=0}^{\infty} (\varepsilon\,\varkappa\,\overline{\mu})^n\,,\ \varepsilon \in [0,\, 1]\,. \tag{3.5}$$

Since $\|\varkappa\|_\infty = k < 1$,

$$\|\varkappa_\varepsilon\|_\infty \le \frac{\varepsilon k}{1 - \varepsilon k} \le \frac{k}{2-k} = q < 1\,, \qquad \text{for } \varepsilon \in [0,\, 1/2]\,. \tag{3.6}$$

Now, let

$$\gamma_\varepsilon(w) := \begin{cases} \left(\varkappa_\varepsilon \cdot \frac{\overline{f_z}}{f_z}\right) \circ f^{-1}(w)\,, & w \in f(B)\,, \\ 0, & w \in f(\mathbb{C} \setminus B)\,. \end{cases} \tag{3.7}$$

Re-defining, in the case of necessity, $\varkappa$ in a set of measure zero, without loss of generality, we may assume that $|\varkappa(z)| \le k$ and $|\varkappa_\varepsilon(z)| \le q$ for all $z \in \mathbb{C}$ and, thus, $\gamma_\varepsilon(z) \le q$ also for all $z \in \mathbb{C}$. Moreover, since $|\mathbb{C} \setminus B| = 0$,

$$\gamma_\varepsilon \circ f = \varkappa_\varepsilon \cdot \frac{f_z}{\overline{f_z}} \quad \text{a.e.} \tag{3.8}$$

Consider the family of Q–quasiconformal ($Q = (1+q)/(1-q)$) mappings $g_\varepsilon : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $\varepsilon \in [0,\, 1/2]$ with the complex characteristics γ_ε, $\varepsilon \in [0,\, 1/2]$ and the normalization $g_\varepsilon(0) = 0$, $g_\varepsilon(1) = 1$ and $g_\varepsilon(\infty) = \infty$, see the existence theorem for quasiconformal mappings, e.g., in the book [1], p. 98. By the theorem on differentiability of Q–q.c. mappings in a parameter (see [1], p. 105):

$$g_\varepsilon(w') = w' - \frac{\varepsilon}{\pi} \int\limits_{f(B)} \gamma(w)\varphi(w,\, w')\, dm_w + o(\varepsilon,\, w') \tag{3.9}$$

where $o(\varepsilon,\, w')/\varepsilon \to 0$ as $\varepsilon \to 0$ loccally uniform with respect to $w' \in \mathbb{C}$ and

$$\gamma(w) = \begin{cases} \left(\varkappa \cdot \frac{f_z}{\overline{f_z}}\right) \circ f^{-1}(w)\,, & w \in f(B)\,, \\ 0, & w \in f(\mathbb{C} \setminus B)\,. \end{cases} \tag{3.10}$$

Next, consider the family of mappings $f_\varepsilon = g_\varepsilon \circ f$, $\varepsilon \in [0,\, 1/2]$. Let us show that $f_\varepsilon \in H^*_M$. First, by Corollary 2.1, $f_\varepsilon \in W^{1,1}_{loc}$. Then note that the regular homeomorphism f has N^{-1}–property by the Ponomarev theorem, see [19]. Hence, similarly to IC(6) in [1], since $J_f(z) \ne 0$ a.e. and $f_z \ne 0$ a.e. we obtain that a.e.

$$\mu_{g_\varepsilon} \circ f = \frac{f_z}{\overline{f_z}} \cdot \frac{\mu_{f_\varepsilon} - \mu_f}{1 - \overline{\mu_f} \cdot \mu_{f_\varepsilon}}\,. \tag{3.11}$$

Here we applied the rule of differentiability of composition (2.6), see Corollary 2.2. Solving (3.11) with respect to μ_{f_ε}, we conclude that a.e.

$$\mu_{f_\varepsilon} = \frac{\mu_{g_\varepsilon} \circ f + \frac{f_z}{\overline{f_z}} \cdot \mu_f}{\frac{f_z}{\overline{f_z}} + \overline{\mu_f} \cdot \mu_{g_\varepsilon} \circ f} = \frac{\mu + \frac{\overline{f_z}}{f_z} \cdot \gamma_\varepsilon \circ f}{1 + \overline{\mu} \cdot \frac{\overline{f_z}}{f_z} \cdot \gamma_\varepsilon \circ f}\,. \tag{3.12}$$

Putting in (3.12) the expressions from (3.5) and (3.8), we have that a.e.

$$\mu_{f_\varepsilon} = \frac{\mu + \varkappa_\varepsilon}{1 + \overline{\mu}\varkappa_\varepsilon} = \frac{\mu + \frac{\varepsilon\varkappa}{1 - \varepsilon\varkappa\overline{\mu}}}{1 + \overline{\mu} \cdot \frac{\varepsilon\varkappa}{1 - \varepsilon\varkappa\overline{\mu}}} = \mu + \varepsilon\varkappa\left(1 - |\mu|^2\right)\,. \tag{3.13}$$

By (3.13) and (3.1) we obtain that $\mu_{f_\varepsilon} = \mu_\varepsilon$ where μ_ε is given by (3.2). Thus, $\mu_{f_\varepsilon} \in \mathfrak{M}_M$, $\varepsilon \in [0,\, 1/2]$ in view of convexity of $\mathfrak{M}_M$.

Note that the homeomorphism f_ε is regular for $\varepsilon \in [0,\, 1/2]$. Indeed, let us assume that f_ε is not regular for some $\varepsilon \in [0,\, 1/2]$. Since $|\mu_{f_\varepsilon}| < 1$ a.e., that would be meant that $(f_\varepsilon)_z = 0 = (f_\varepsilon)_{\overline{z}}$ on a set $E \subseteq \mathbb{C}$ of a positive measure where f_ε is differentiable and f is regular. Then similarly to IC(2) in [1], we obtain that everywhere on E

$$(g_\varepsilon)_w \circ f = \frac{1}{J_f} \left[(f_\varepsilon)_z \overline{f_z} - (f_\varepsilon)_{\overline{z}}\, \overline{f_{\overline{z}}} \right] = 0\,, \tag{3.14}$$

see Proposition 2.2. However, the set $\mathcal{E} := f(E)$ has measure zero because g_ε is a quasiconformal mapping. Thus, we come to the contradiction with the N^{-1}–property of the mapping f, see [19]. Consequently, $f_\varepsilon \in H^*_M$, $\varepsilon \in [0,\, 1/2]$.

Finally, changing variables in (3.9), we come to (3.3) because $|\mathbb{C} \setminus B| = 0$.

4 Variational maximum principle

They say that a functional $\Omega : H^*_M \to \mathbb{R}$ is **differentiable by Gateaux** if

$$\Omega(f_\varepsilon) = \Omega(f) + \varepsilon Re \int\limits_{\mathbb{C}} g\, d\varkappa + o(\varepsilon) \tag{4.1}$$

for every variation $f_\varepsilon = f + \varepsilon g + o(\varepsilon)$ in the class H^*_M where $\varkappa = \varkappa_f$ is a finite comlex Radon measure with a compact support and $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$ locally uniform in $\mathbb{C}$ (see [27], pp. 138–139). In other words, there is a continuous linear functional $L(g; f)$ in the first variable such that

$$\Omega(f_\varepsilon) = \Omega(f) + \varepsilon\, Re\, L(g; f) + o(\varepsilon)\,. \tag{4.2}$$

Later we assume that the function $\varphi(w, f(\zeta))$ is locally integrable for every $f \in H^*_M$ with respect to the product of measures $dm_w \otimes d\varkappa(\zeta)$ where φ is the kernel from (3.4), m is the Lebesgue measure in $\mathbb{C}$ and that

$$A(w) = \frac{1}{\pi} \int\limits_{\mathbb{C}} \varphi(w, f(\zeta))\, d\varkappa(\zeta) \neq 0 \quad \text{for a.e. } w \in \mathbb{C}\,. \tag{4.3}$$

Then we say that Ω is differentiable by Gateaux **without degeneration** on the class H^*_M.

Theorem 4.1. *Let $M(z)$, $z \in \mathbb{C}$, be a family of compact convex sets in $\mathbb{D}$ which is measurable in the parameter z such that $Q_M \in L^1_{loc}$ and let a functional $\Omega : H^*_M \to \mathbb{R}$ is differentiable by Gateaux without degeneration. If $\max \Omega$ in the class H^*_M is attained for a mapping f, then its complex characteristic satisfies the inclusion*

$$\mu(z) \in \partial M(z) \quad \textit{for a.e. } z \in \mathbb{C}\,. \tag{4.4}$$

Proof. Since $\mu \in \mathfrak{M}_M$, without loss of generality we may assume that $\mu(z) \in M(z)$ for all $z \in \mathbb{C}$. Let us assume that the set

$$E = \{z \in \mathbb{C} : \mu(z) \notin \partial M(z)\}$$

has a positive Lebesgue measure. Let

$$E_m = \{z \in \mathbb{C} : Q_M(z) \leq m\}, \quad m = 1, 2, \ldots,$$

$$K(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}, \quad z_0 \in \mathbb{C},\ r > 0\,,$$

$\chi,\ \chi_m,\ \chi_{z_0,r}$ are characteristic functions of the sets $E,\ E_m,\ K(z_0, r)$, respectively. Now, let $\alpha_n,\ n = 1, 2, \ldots,$ be an enumeration of all rational numbers in $[0, 2\pi)$ and $\rho_n(z),\ n = 1, 2, \ldots,$ be distances from $\mu(z)$ till the points of intersections of rays $\mu(z) + te^{i\alpha_n},\ t > 0$, with $\partial M(z)$.

Let us show that the functions $\rho_n(z),\ n = 1, 2, \ldots$, are measurable in the parameter z. Indeed, let $\Lambda_n(z) = \{\nu \in \mathbb{C} : \nu = \mu(z) + te^{i\alpha_n}, \quad 0 \leq t \leq 2\}$ be the segment of the ray passing from the point $\mu(z)$ in the direction $e^{i\alpha_n}$ of the length 2. The measurability of the families of the sets $\Lambda_n(z)$ in z follows, e.g., from Proposition 3.1 in [21] and general properties of elementary operations with measurable functions (see, e.g., [25]). Consequently, the families $M_n(z) = M(z) \cap \Lambda_n(z)$ and $\{\eta_n(z)\} = \partial\mathbb{D} \cap \Lambda_n(z)$ where $\partial\mathbb{D} = \{\eta \in \mathbb{C} : |\eta| = 1\}$ is the unit circle are also measurable (see Lemma 3.3 in [21]). Thus, the functions $\eta_n(z),\ n = 1, 2, \ldots,$ are measurable, e.g., by the criterion 6) in Proposition 6 in [20]. By Proposition 3.1 in [21] the distance functions $r_n(z) = \min\limits_{\nu \in M_n(z)} |\nu - \eta_n(z)|$ are also measurable. It remains to note after this that $\rho_n(z) = |\mu(z) - \eta_n(z)| - r_n(z)$.

Next, consider the functions $\mu_n(z) = \mu(z) + \rho_n(z)e^{i\alpha_n}$. By construction they belong to the class $\mathfrak{M}_M$. Since the sets $M(z)$ are convex, the functions

$$\nu_n(z) := \mu(z) + \lambda(z)(\mu_n(z) - \mu(z)) = (1 - \lambda(z))\mu(z) + \lambda(z)\mu_n(z)$$

also belong to the class $\mathfrak{M}_M$ for an arbitrary measurable function $\lambda(z) : \mathbb{C} \to [0, 1]$. In particular, the class $\mathfrak{M}_M$ contains the functions

$$\nu^{m,n}_{z_0,r}(z) := \mu(z) + \lambda_m(z)\chi_{z_0,r}(z)(\mu_n(z) - \mu(z))$$

where

$$\lambda_m(z) = \frac{1 - |\mu(z)|^2}{2}\chi(z)\chi_m(z)\,.$$

Note that

$$|\mu_n(z) - \mu(z)| = \rho_n(z) \leq 2q_M(z)$$

and

$$\varkappa^{m,n}_{z_0,r}(z) := \frac{\nu^{m,n}_{z_0,r}(z)-\mu(z)}{1-|\mu(z)|^2} = \frac{\mu_n(z)-\mu(z)}{2}\chi(z)\chi_m(z)\chi_{z_0,r}(z)$$

belong to the closed ball of the radius $q_m := (m-1)/(m+1) < 1$ in $L^\infty(\mathbb{C})$.

Since f is extremal, applying the variation of Theorem 3.1 with $\nu = \nu^{m,n}_{z_0,r}$, we obtain that

$$Re \int\limits_{\mathbb{C}} \left[\ \int\limits_{|z-z_0|\le r} \varphi_{m,n}(z,\zeta)\, dm_z\right] d\varkappa(\zeta) \ge 0 \tag{4.5}$$

where

$$\varphi_{m,n}(z,\zeta) = \lambda_m(z)(\mu_n(z)-\mu(z))f_z^2\varphi(f(z),f(\zeta))\,.$$

Consider the functions

$$\psi^{m,n}_{z_0,r}(w,\zeta) = \begin{cases} \left(\varkappa^{m,n}_{z_0,r}\cdot\frac{f_z}{\overline{f_z}}\right)\circ f^{-1}(w)\ \varphi(w,f(\zeta))\,, & w\in f(B)\,,\\ 0, & w\in f(\mathbb{C}\setminus B)\,, \end{cases}$$

where B denotes the (Borel) set of all points in $\mathbb{C}$ where the mapping f has a total differential and $J_f(z)\ne 0$. They are integrable with respect to the product of the measures $dm_w\otimes d\varkappa(\zeta)$. Note that

$$J_{f^{-1}}(w) = \left[J_f\left(f^{-1}(w)\right)\right]^{-1} = \left[(1-|\mu|^2)f_z^2\right]^{-1}\left(f^{-1}(w)\right)$$

at every point $w\in f(B)$, cf. IC(3) in [1]. Moreover, since the regular homeomorphism f has N^{-1}–property, after the replacement of variables (see Lemmas III.2.1 and III.3.2 in [15]) we obtain that the functions $\varphi_{m,n}(z,\zeta)$ are also integrable with respect to the measure product $dm_z\otimes d\varkappa(\zeta)$ and by the Lebesgue–Fubini theorem (see, e.g., Theorem V.8.1 in [4]) and (4.5) we conclude that

$$\int\limits_{|z-z_0|\le r}\left[Re\int\limits_{\mathbb{C}}\varphi_{mn}(z,\zeta)\,d\varkappa(\zeta)\right]dm_z \ge 0\,.$$

By the Lebesgue theorem on the differentiability of the indefinite integral (see, e.g., Theorem IV(5.4) in [25]) we have the inequalities

$$\lambda_m(z)Re(\mu_n(z)-\mu(z))\mathcal{B}(z)\ge 0 \quad \text{for a.e. } z\in\mathbb{C},\ \ m,\, n=1,\,2,\,\ldots,$$

where $\mathcal{B}(z) = \mathcal{A}(f(z))f_z^2$ and $\mathcal{A}(w)$ is given by (4.3). Hence

$$\rho_n(z)Re\,\mathcal{B}(z)e^{i\alpha_n}\ge 0,\quad n=1,\,2,\,\ldots,\quad \text{for a.e. } z\in E\cap E_m\,.$$

Since $E_m,\ m=1,\,2,\,\ldots$, form an exhaustion of the plane $\mathbb{C}$ in measure, the last holds for a.e. $z\in\mathbb{C}$. On the other hand $\rho_n(z)>0,\ n=1,\,2,\,\ldots$, on E and, thus, this is equivalent to the inequalities

$$Re\,\mathcal{B}(z)e^{i\alpha_n}\geq 0\,,n=1,\,2,\,\ldots,\quad \text{for a.e. } z\in E\,.$$

By arbitrariness of $\alpha_n,\ n=1,\,2,\,\ldots$, we have from here that

$$Re\,\mathcal{B}(z)e^{i\alpha}\geq 0\quad \forall\,\alpha\in[0,2\pi)\quad \text{for a.e. } z\in E\,.$$

In particular, for $\alpha=0$ and $\alpha=\pi$ we obtain that $\pm Re\,\mathcal{B}(z)\geq 0$, a.e. $Re\,\mathcal{B}(z)=0$, and for $\alpha=\pi/2$ and $\alpha=3\pi/2$: $\pm Im\,\mathcal{B}(z)\geq 0$, i.e. $Im\,\mathcal{B}(z)=0$. Thus, $\mathcal{B}(z)=0$ for a.e. $z\in E$. However, the latter is impossible because $\mathcal{A}(w)\neq 0$ a.e., f has N^{-1}–property and $f_z\neq 0$ a.e. The obtained contradiction shows that $mesE=0$, i.e. $\mu(z)\in\partial M(z)$ a.e.

5 Other necessary conditions for extremum

To formulate the necessary conditions of the extremum we need one more notion. Namely, let $\mu\in\mathfrak{M}_M$. Then $\omega_\mu(z)$ denotes the **cone of the admissible directions** (see, e.g., [14]) for the set $M(z)$ at the point $\mu(z)$, a.e., the set of all $\omega\in\mathbb{C},\ \omega\neq 0$, such that $\mu(z)+\varepsilon\omega\in M(z)$ for all $\varepsilon\in[0,\varepsilon_0]$ and some $\varepsilon_0>0$. Note that for strictly convex sets $M(z)$, being invariant–convex sets, the cone of admissible directions $\omega_\mu(z)$ is an open cone for every z. Almost word for word repeating the proof of Theorem 4.1, we obtain:

Theorem 5.1. *Under the hypothesis of Theorem 4.1, the extremal f in the problem on $\max\Omega$ in the class H_M^* satisfies the inequalities*

$$Re\,\omega\mathcal{B}(z)\geq 0 \tag{5.1}$$

for a.e. $z\in\mathbb{C}$ for all ω in the cone of admissible directions $\omega_\mu(z)$ where $\mathcal{B}(z)=\mathcal{A}(f(z))f_z^2$ and $A(w)$ is given by (4.3).

Corollary 5.1. *If in addition, the boundary is regular for a.e. $z\in\mathbb{C}$, i.e., $\partial M(z)$ has a tangent at every point, then (5.1) is transformed to the inequality*

$$n(z)\mathcal{B}(z)\geq 0\quad a.e. \tag{5.2}$$

where $n(z)$ is the unit vector of the inner normal to $\partial M(z)$ at the point $\mu(z)$.

In particular, if $M(z)$ is a family of disks

$$M(z)=\{\varkappa\in\mathbb{C}:|\varkappa-c(z)|\leq k(z)\} \tag{5.3}$$

where the functions $c(z)$ and $k(z)$ are measurable, then by the maximum principle, Theorem 4.1, $n(z) = (c(z) - \mu(z))/k(z)$ and the relation (5.2) is equivalent to the equality

$$\frac{c(z) - \mu(z)}{k(z)} = \frac{\overline{\mathcal{B}(z)}}{|\mathcal{B}(z)|} \quad \text{a.e.}\,,$$

i.e.,

$$\mu(z) = c(z) - k(z)\frac{\overline{\mathcal{B}(z)}}{|\mathcal{B}(z)|}\,.$$

Thus, we have:

Corollary 5.2. *Let $M(z)$, $z \in \mathbb{C}$, be the family of the disks (5.3), $k + |c| \in L^1_{loc}$, and the functional $\Omega : H^*_M \to \mathbb{R}$ is differentiable by Gateaux without degeneration. Then the extremal of the problem on $\max \Omega$ in the class H^*_M satisfies the equality*

$$f_{\overline{z}} = c(z) f_z - k(z)\frac{\overline{\mathcal{A}(f(z))}}{|\mathcal{A}(f(z))|}\,\overline{f_z}\,. \tag{5.4}$$

In particular, if $c(z) = 0$ we obtain the equality

$$f_{\overline{z}} = -k(z)\,\frac{\overline{\mathcal{A}(f(z))}}{|\mathcal{A}(f(z))|}\,\overline{f_z}\,. \tag{5.5}$$

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